

Perturbative QCD at non-zero chemical potential: Comparison with the large- N_f limit and apparent convergence

A. Ipp and A. Rebhan

*Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Haupstr. 8-10, A-1040 Vienna, Austria*

A. Vuorinen

*Department of Physical Sciences and Helsinki Institute of Physics
P.O. Box 64, FIN-00014 University of Helsinki, Finland*

The perturbative three-loop result for the thermodynamic potential of QCD at finite temperature and chemical potential as obtained in the framework of dimensional reduction is compared with the exact result in the limit of large flavor number. The apparent convergence of the former as well as possibilities for optimization are investigated. Corresponding optimized results for full QCD are given for the case of two massless quark flavors.

PACS numbers: 11.10.Wx, 12.38.Mh, 11.15.Pg

I. INTRODUCTION

Quite some progress has been made recently in extending perturbative and nonperturbative results on the thermodynamic potential of hot deconfined QCD to nonzero chemical potential. On the lattice, where a nonzero quark chemical potential gives rise to a computationally problematic complex action, a number of new techniques have been devised that make it possible to study the effects of a not too large quark chemical potential $\mu \lesssim T$, and first results have been produced for the equation of state [1, 2, 3]. Concurrently, the perturbative series expansion of the QCD pressure, which had been determined up to and including the last fully perturbative order $g^6 \ln g$ in Refs. [4, 5, 6, 7], has been extended to non-zero chemical potential in Refs. [8, 9, 10] within the framework of dimensional reduction [6, 11]. Dimensional reduction can be expected to work as long as $2\pi T$ is the dominant energy scale. In the limit of large μ/T this ceases to be the case when the scale $g\mu$ appearing in the three-dimensional mass parameter m_E becomes of the same order. In the regime $T \lesssim g\mu$, perturbation theory requires a reorganization with new phenomena such as the onset of non-Fermi-liquid behavior [12], and for the parametrically smaller temperatures $T \lesssim \mu g^{-5} \exp[-3\pi^2/(\sqrt{2}g)]$ eventually the nonperturbative phenomenon of color superconductivity arises [13].

Since at any temperature of practical or even theoretical relevance, the QCD coupling g is not much smaller than 1, the predictivity of the perturbative results remains uncertain. In fact, it is a well known problem of perturbative results at finite temperature that the strictly perturbative series expansion has extremely poor apparent convergence and large renormalization scale dependence up to temperatures of at least 10^5 times the deconfinement temperature T_c [5, 6]. On the other hand, it has been found that already resummations involving only the lowest order contributions from the soft scale gT can improve the situation decisively ([14] and references

therein). Within the framework of dimensional reduction, a significant improvement can be achieved by the simple resummation provided by keeping effective-field-theory parameters unexpanded [7, 15].

In particular at non-zero quark chemical potential, where lattice data are still somewhat preliminary, it would be helpful to be able to test the predictivity of the (resummed) perturbative results by other means. Most recently the limit of large flavor number $N_f \rightarrow \infty$, with finite $g_{\text{eff}}^2 \equiv g^2 N_f/2$, has been developed as such a theoretical test bed. In Refs. [16] the large- N_f pressure has been determined exactly for all temperatures sufficiently below the scale of the Landau pole $\Lambda_L \sim T \exp(6\pi^2/g_{\text{eff}}^2)$ that is present in the large- N_f theory; in Ref. [17] this has now been extended to non-zero chemical potential. In this Brief Report, we use the results of Ref. [17] obtained in the large- N_f limit to investigate the apparent convergence of the perturbative results of Ref. [10] obtained in dimensional reduction at non-zero chemical potential. We are also able to study for the first time quantitatively the inevitable breakdown of dimensional reduction at large μ/T in the context of the pressure. For moderate μ/T , we consider the possibilities for optimization of the perturbative results, and present correspondingly optimized results for $N_f = 2$ massless flavors in full QCD.

II. LARGE N_f

In the limit of a large number of quark flavors N_f with a common chemical potential μ , the dimensional reduction result for the interaction part of the pressure of QCD at finite temperature and chemical potential obtained in [10] reduces to ($g_{\text{eff}}^2 \equiv g^2 N_f/2$)

$$\begin{aligned} \frac{P - P_0}{N_g} \bigg|_{N_f \rightarrow \infty}^{\text{DR}} = & - \left(\frac{5}{9} T^4 + \frac{2}{\pi^2} \mu^2 T^2 + \frac{\mu^4}{\pi^4} \right) \frac{g_{\text{eff}}^2}{32} \\ & + \frac{1}{12\pi} m_E^3 T + \bar{\alpha}_{E3} \frac{g_{\text{eff}}^4}{(48\pi)^2} + O(g_{\text{eff}}^6), \end{aligned} \quad (1)$$

where N_g is the number of gluons. The effective field theory parameter m_E^2 is given by

$$m_E^2 = \left(T^2 + \frac{3\mu^2}{\pi^2} \right) \left[\frac{g_{\text{eff}}^2}{3} - \frac{g_{\text{eff}}^4}{(6\pi)^2} \left(2 \ln \frac{\bar{\mu}_{\text{MS}}}{4\pi T} - 1 - \aleph(z) \right) \right] + O(g_{\text{eff}}^6), \quad (2)$$

and the coefficient of the order g_{eff}^4 -term in (1) by

$$\begin{aligned} \bar{\alpha}_{E3} = & 12 \left\{ \frac{5}{9} T^4 + \frac{2}{\pi^2} \mu^2 T^2 + \frac{\mu^4}{\pi^4} \right\} \ln \frac{\bar{\mu}_{\text{MS}}}{4\pi T} \\ & + 4T^4 \left\{ \frac{1}{12} + \gamma - \frac{16}{15} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{8}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right\} \\ & + \frac{\mu^2 T^2}{\pi^2} \left\{ 14 + 24\gamma - 32 \frac{\zeta'(-1)}{\zeta(-1)} \right\} + \frac{\mu^4}{\pi^4} (43 + 36\gamma) \\ & - 96T^4 \{ 3\aleph(3, 1) + 8\aleph(3, z) + 3\aleph(3, 2z) - 2\aleph(1, z) \} \\ & + \frac{48i\mu T^3}{\pi} \{ \aleph(0, z) - 12\aleph(2, z) - 12\aleph(2, 2z) \} \\ & + \frac{96\mu^2 T^2}{\pi^2} \{ 4\aleph(1, z) + 3\aleph(1, 2z) \} \\ & + \frac{144i\mu^3 T}{\pi^3} \aleph(0, z) \end{aligned} \quad (3)$$

with $z \equiv \frac{1}{2} - i\frac{\mu}{2\pi T}$ and the special functions [10]

$$\aleph(n, w) \equiv \zeta'(-n, w) + (-1)^{n+1} \zeta'(-n, w^*), \quad (4)$$

$$\aleph(w) \equiv \Psi(w) + \Psi(w^*). \quad (5)$$

ζ is the Riemann zeta function, $\zeta'(x, y) \equiv \partial_x \zeta(x, y)$, and Ψ the digamma function $\Psi(w) \equiv \Gamma'(w)/\Gamma(w)$. Note that despite the appearance of complex quantities in (3) the coefficient $\bar{\alpha}_{E3}$ is real as it has to be.

To the order of accuracy of the result (1), the relevant dimensionally reduced effective theory in the large- N_f limit turns out to be noninteracting, contributing only the term $m_E^3 T$ in (1). This gives rise to nonanalytic terms in g_{eff}^2 , namely single powers of g_{eff} , but no logarithmic terms as in full QCD.

Just as in full QCD, however, the perturbative result has unusually large renormalization scale dependence for $g_{\text{eff}} \gtrsim 2$. But, as noted in Ref. [16], choosing the renormalization scale $\bar{\mu}_{\text{MS}}$ such that the g_{eff}^4 correction in m_E^2 as given in Eq. (2) is put to zero (FAC-m) gives good agreement with the exact result up to $g_{\text{eff}}^2 \sim 7$ at zero chemical potential. At large N_f , this prescription is also FAC with respect to the pressure itself, since it puts to zero the coefficient of its g_{eff}^5 -term.

In the upper panel of Fig. 1 the quality of the perturbative result to order g_{eff}^5 in the large- N_f limit with $\bar{\mu}_{\text{MS}} = \bar{\mu}_{\text{MS}}^{\text{FAC-m}}$ is displayed for the entire μ - T plane. The deviation of the thus optimized perturbative result for the interaction part of the pressure from the exact one is shown in the form of a contour plot of $[(P_{\text{DR}} - P_0) - (P - P_0)]/|P - P_0| = (P_{\text{DR}} - P)/|P - P_0|$, where P is the exact result from Ref. [17] and P_0 the

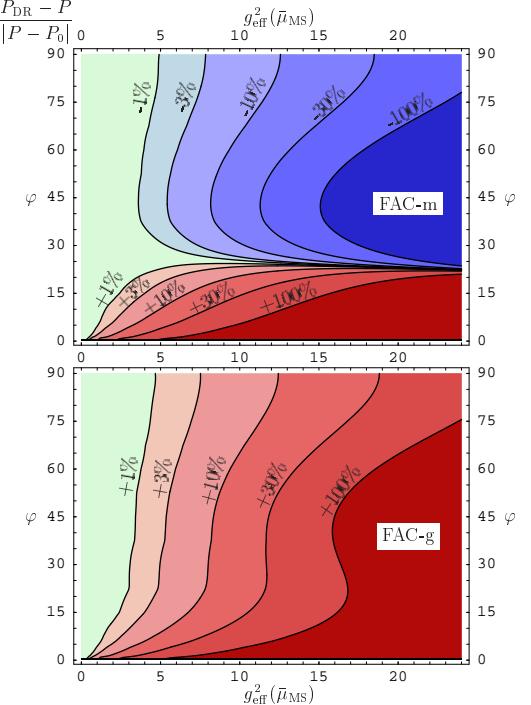


FIG. 1: Percentage errors of the perturbative result for the interaction part of the pressure to order g_{eff}^5 in the large- N_f limit for two choices of $\bar{\mu}_{\text{MS}}$: Fastest apparent convergence of P as well as m_E^2 (FAC-m), and of g_E^2 (FAC-g). The brightest area corresponds to an error of less than 1%, the darkest ones to an error of over 100%. The ratio of chemical potential to temperature increases from top to bottom according to $\varphi = \arctan(\pi T/\mu)$ so that 90° corresponds to high temperature and zero chemical potential, and 0° to zero temperature and high chemical potential. The coupling is given in terms of $g_{\text{eff}}^2(\bar{\mu}_{\text{MS}})$ at $\bar{\mu}_{\text{MS}} = \sqrt{\pi^2 T^2 + \mu^2}$.

ideal-gas value. The μ - T plane is parametrized by $\varphi = \arctan(\pi T/\mu)$ and $g_{\text{eff}}^2(\bar{\mu}_{\text{MS}})$ at $\bar{\mu}_{\text{MS}} = \sqrt{\pi^2 T^2 + \mu^2}$. The accuracy of the FAC-m result at zero chemical potential ($\varphi = 90^\circ$) is seen to decrease slowly with increasing chemical potential, apart from an accidental zero of the error when $\varphi \sim 22^\circ$; for slightly smaller $\varphi \lesssim 18^\circ$, i.e. $T \lesssim 0.1\mu$, the errors eventually start to grow rapidly, marking the breakdown of dimensional reduction.

While FAC-m is particularly natural in the large- N_f limit, one can consider different optimizations. At sub-leading orders in the large- N_f expansion, one would also need the effective-field-theory parameter

$$g_E^2 = T \left[g_{\text{eff}}^2 + \frac{g_{\text{eff}}^4}{3(2\pi)^2} \left(2 \ln \frac{\bar{\mu}_{\text{MS}}}{4\pi T} - \aleph(z) \right) \right] + O(g_{\text{eff}}^6) \quad (6)$$

and one could choose to set $\bar{\mu}_{\text{MS}}$ such that the g_{eff}^4 correction in g_E^2 is put to zero (FAC-g). Curiously enough, the corresponding $\bar{\mu}_{\text{MS}}$ happens to be strictly proportional to that of the FAC-m scheme, $\bar{\mu}_{\text{MS}}^{\text{FAC-g}}(T, \mu) = e^{-1/2} \bar{\mu}_{\text{MS}}^{\text{FAC-m}}(T, \mu)$. In contrast to FAC-m, this does not minimize the g_{eff}^5 coefficient in the pressure. Neverthe-

less, as can be seen in the lower panel of Fig. 1, the quality of the perturbative result is rather similar to FAC-m: the sign of the deviation is now positive throughout the μ/T plane so that there is no accidental zero of the error, but the absolute magnitude of the error is otherwise comparable.

Because FAC-m and FAC-g have comparable errors with opposite sign for $\varphi \gtrsim 18^\circ$, the optimal choice of renormalization scale for the 3-loop result happens to lie in between the two FAC scales. Incidentally, their arithmetic mean turns out to give a surprising accuracy with errors below 1% for all values of g_{eff}^2 considered, if $\varphi \gtrsim 67^\circ$ or $\mu/T \lesssim 1.3$. However, for larger μ/T the range of coupling with small errors shrinks quickly; for $T \lesssim 0.1\mu$ the errors are finally no smaller than with either FAC-m or FAC-g.

Thus, except at rather small T/μ , the two FAC optimizations that we have considered appear to give very satisfactory results in the large- N_f limit, even at coupling g_{eff}^2 so large that the renormalization scale dependence is already quite strong. The renormalization scales they lead to are in fact rather close to each other—they differ just by a factor of $e^{1/2} \approx 1.6$, and they (perhaps fortuitously) turn out to enclose the actual optimal choice for $T/\mu \gg 0.1$.

III. FINITE N_f

It is of course far from guaranteed that the optimizations FAC-m and FAC-g will work equally well in full, i.e. finite- N_f QCD (in the deconfinement phase). When applying these prescriptions now to the case of $N_f = 2$, we shall therefore also consider the size of the renormalization scale dependence as a further criterion and investigate whether it can be reduced by a simple resummation consisting of keeping effective-field-theory parameters unexpanded.

In Ref. [15], it has been found that at zero chemical potential such a resummation considerably reduces the scale dependence. Using a standard 2-loop running coupling one can even fix the renormalization scale by a principle of minimal sensitivity (PMS), i.e. by requiring that the derivative of P with respect to $\bar{\mu}_{\text{MS}}$ vanishes, and this turns out to be close to the FAC result.¹ While the formal $\bar{\mu}_{\text{MS}}$ -independence of the perturbative result to order g^5 or $g^6 \ln g$ requires only a one-loop beta function, it is not inconsistent to use a more accurate coupling. We in particular adopt the 2-loop running coupling because the QCD scale Λ_{QCD} of other renormalization schemes is frequently related to the one of the $\overline{\text{MS}}$ scheme using the 2-loop beta function. Also, the 2-loop coupling is already reasonably close to the 3- or 4-loop coupling.

¹ In the large- N_f limit PMS cannot be applied because there the scale dependence turns out to be monotonic.

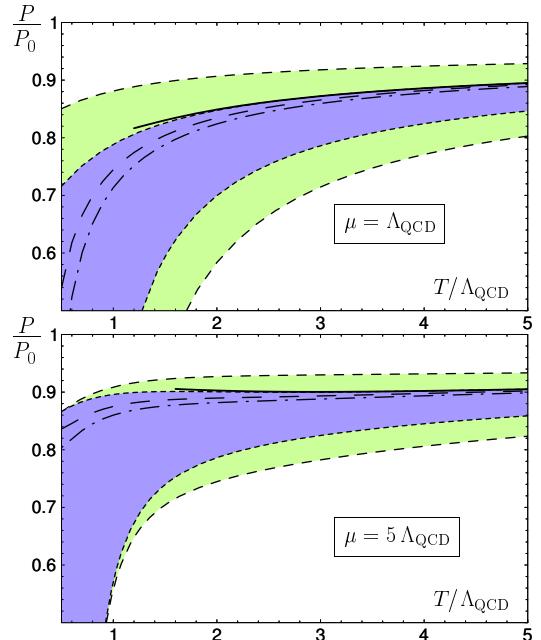


FIG. 2: Three-loop pressure in deconfined QCD with two massless flavors for $\mu = \Lambda_{\text{QCD}}$ (upper panel) and $\mu = 5\Lambda_{\text{QCD}}$ (lower panel). The bright-colored bands show the strictly perturbative results to order g^5 with $\bar{\mu}_{\text{MS}}$ varied about the FAC-m value (dash-dotted line) by a factor of 2, the darker bands the same when m_E^2 is left unexpanded. The dashed line is the FAC-g result, the full line beginning at $T \sim 1.5\Lambda_{\text{QCD}}$ the PMS result.

In Fig. 2 we have repeated the analysis of the scale dependence of [15] for the perturbative $N_f = 2$ result of Ref. [10] to order² g^5 with a quark chemical potential of $\mu = \Lambda_{\text{QCD}}$ as well as $\mu = 5\Lambda_{\text{QCD}}$. The former value is within what is presently being achieved in lattice calculations for $N_f = 2$ [3]; for $\mu = 5\Lambda_{\text{QCD}}$ one may expect to be in a quark-gluon phase for all values of the temperature. In the plots of Fig. 2, the temperature is varied in both cases between 0.5 and 5 times Λ_{QCD} .

The case $\mu = \Lambda_{\text{QCD}}$ turns out to be qualitatively similar to the $\mu = 0$ case studied in [15]: the scale dependence is greatly reduced by keeping m_E^2 unexpanded, and the optimization using FAC is close to PMS (where the latter exists). For $T \lesssim \Lambda_{\text{QCD}}$ the scale dependence increases strongly so that one probably should conclude that any predictivity is lost. On the other hand, all optimized results are very close to each other for $T \gtrsim 1.5\Lambda_{\text{QCD}}$ and may be considered as rather definite predictions.

In the case $\mu = 5\Lambda_{\text{QCD}}$, the situation is similar for $T \gtrsim 1.5\Lambda_{\text{QCD}}$, but the scale dependences turn out to increase more rapidly as T is lowered; moreover, keep-

² We refrain, however, from extending this analysis to the order $g^6 \ln g$ contribution as done in Ref. [15] for zero chemical potential, because it would involve an unknown function of μ/T rather than a mere constant.

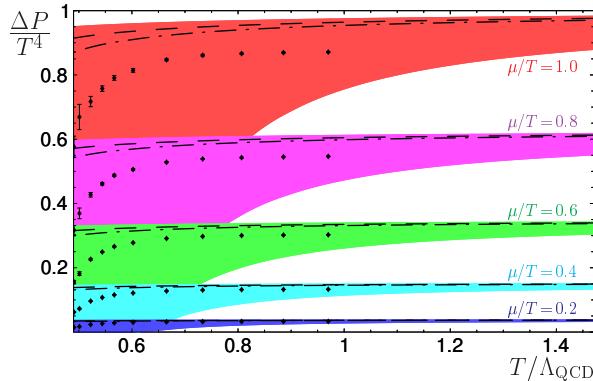


FIG. 3: The difference $\Delta P = P(T, \mu) - P(T, 0)$ divided by T^4 using the unexpanded three-loop result from dimensional reduction of Ref. [10] for $\mu/T = 0.2, \dots, 1.0$ (bottom to top). Shaded areas correspond to a variation of $\bar{\mu}_{\text{MS}}$ around the FAC-m choice by a factor of 2; dashed and dash-dotted lines correspond to the FAC-g and FAC-m results, respectively. Also included are the recent lattice results of Ref. [3] (not yet continuum-extrapolated!) assuming $T_c^{\mu=0} = 0.49 \Lambda_{\text{QCD}}$ [18].

ing m_E^2 unexpanded does no longer help to improve this. Yet, the two FAC results are fairly close to each other for $T \gtrsim 0.5\Lambda_{\text{QCD}}$, but show a run-away behavior for T slightly smaller than $0.5\Lambda_{\text{QCD}}$ (not displayed). This could indicate a beginning breakdown of dimensional reduction, or at least of its predictivity. On the other hand, the FAC results appear to remain predictive for $T \gtrsim 1.5\Lambda_{\text{QCD}}$ even when $\mu \sim 5\Lambda_{\text{QCD}}$. At much higher μ , however, T needs to be correspondingly higher in order that reasonable FAC results are obtained, but this is primarily due to the run-away behavior of the FAC scales and not so much due to the behavior of the pressure itself.

Another popular choice of renormalization scales is the BLM prescription [19], which amounts to simply keeping the FAC scales as obtained in the $N_f \rightarrow \infty$ limit. These

scales show no run-away behaviour, but are significantly smaller than those of FAC-g and FAC-m at finite N_f , with results roughly at the lower boundaries of the bands of Fig. 2, i.e. also comparatively far from PMS.

We also note in passing that just as in the $N_f \rightarrow \infty$ limit the two FAC scales happen to be simply proportional to each other when $N_f = 2$: $\bar{\mu}_{\text{MS}}^{\text{FAC-g}} / \bar{\mu}_{\text{MS}}^{\text{FAC-m}} = e^{5/29} \approx 1.188$, while for $N_f \neq 2$ this ratio has nontrivial μ/T -dependence.

In Fig. 3 we display the optimized results for the difference $\Delta P = P(T, \mu) - P(T, 0)$ for various μ/T corresponding to the recent lattice results given in Fig. 6 of Ref. [3] assuming $T_0 \equiv T_c^{\mu=0} = 0.49 \Lambda_{\text{QCD}}$ [18]. At $T/T_0 = 2$, the highest value considered in [3], our FAC-g and FAC-m results exceed the not-yet-continuum-extrapolated lattice data consistently by 10.5% and 9%, respectively. This is in fact roughly the expected discretization error [20]. When normalized to the free value ΔP_0 instead of T^4 , the results would be essentially μ -independent and thus also very similar to the $N_f = 2+1$ lattice results of Ref. [1] as well as the quasiparticle model results of Refs. [21, 22].

To summarize, we have found that the requirement of fastest apparent convergence of effective-field-theory parameters works remarkably well when the perturbative 3-loop result for the pressure of QCD at finite temperature and chemical potential is compared with the exactly solvable large- N_f limit, except for the region $T \lesssim 0.1\mu$. The perturbative results for finite N_f turn out to agree reasonably well with existing lattice data for deconfined QCD with non-zero μ , and also the otherwise strong renormalization scale dependence is brought under control, when effective-field-theory parameters are kept unexpanded as proposed in Refs. [7, 15].

A. Ipp has been supported by the Austrian Science Foundation FWF, project no. 14632-PHY; A. Vuorinen has been supported by the Magnus Ehrnrooth Foundation and the Academy of Finland, contract no. 77744.

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